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ON OPTIMAL STABILIZATION OF CONTROLLED SYSTEMS*

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The motion is investigated of dynamic systems with optimal stabilization in the sense of the method in /1,2/ in the presence of modulus constraints on the controls. The result obtained is applied to the optimal stabilization problem for the stationary motion of a satellite relative to the center of mass, located at a triangular libration point of a two-body system /3/. For a certain class of asymptotically stable systems an estimate is given for the time after which the perturbed motion from an arbitrary point of the attraction domain falls into a prescribed region of the phase space, containing the unperturbed motion.

1. We consider the equations of perturbed motion of the controlled system

$$\frac{dx_s}{dt} = X_s(t, \mathbf{x}) + \sum_{j=1}^{r} m_{sj}(t, \mathbf{x}) u_j{}^\circ(t, \mathbf{x}) \quad (s = 1, \ldots, n) \tag{1.1}$$

Here $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-dimensional real phase vector, $u_1{}^\circ(t, \mathbf{x}), \ldots, u_r{}^\circ(t, \mathbf{x})$ are the controls effecting the optimal stabilization of the unperturbed motion $\mathbf{x} = 0$ in the sense of the method in /1,2/ with control performance index

$$I = \int_{t_0}^{\infty} [F(t, \mathbf{x}[t]) + \Sigma]\, dt, \quad \Sigma = \sum_{i,j=1}^{r} \beta_{ij} u_i u_j \tag{1.2}$$

$$F(t, \mathbf{x}) = -W(t, \mathbf{x}) + \Sigma^\circ, \ \Sigma^\circ = \sum_{i,j=1}^{r} \beta_{ij} u_i{}^\circ u_j{}^\circ$$

Σ is a prescribed positive-definite quadratic form with symmetric coefficients

$$W(t, \mathbf{x}) = dV(t, \mathbf{x})/dt$$

$V(t, \mathbf{x})$ is a postive-definite Liapunov function admitting of an infinitesimal upper bound in the region

$$t \geqslant t_0, \ |x_s| \leqslant l_s, \ l_s = \text{const} > 0 \tag{1.3}$$

The functions $X_s(t, \mathbf{x})$, $m_{sj}(t, \mathbf{x})$, $u_j{}^\circ(t, \mathbf{x})$ are continuous and satisfy conditions ensuring the existence and uniqueness of the solutions of Eqs.(1.1) under any initial conditions from region (1.3). The time derivative of function $V(t, \mathbf{x})$ is taken in virtue of Eqs.(1.1) with $u_j{}^\circ \equiv 0$, and by assumption $W(t, \mathbf{x})$ is a negative-definite or constantly-negative function, while $X_s(t, 0) = 0$.

The optimal controls $u_j{}^\circ$ are determined by the expressions (Δ_{kj} are the cofactors of the elements β_{kj} of determinant Δ) /1/

$$u_j{}^\circ(t, \mathbf{x}) = -\frac{1}{2} \sum_{k=1}^{r} \frac{\Delta_{kj}}{\Delta} \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} m_{ik}, \quad \Delta = \|\beta_{ij}\| > 0 \tag{1.4}$$

When the system being stabilized is autonomous, while the manifold M of points $\mathbf{x}$, defined by the equation

$$W(\mathbf{x}) - 2\Sigma^\circ = 0 \tag{1.5}$$

does not contain integral trajectories of (1.1), except $\mathbf{x} = 0$, then Eqs.(1.4) ensure the asymptotic stability of the unperturbed motion. Expressions (1.4) were obtained without constraints on the magnitude of the controls. If a modulus constraint

$$|u_j{}^\circ| < u_{0j}, \ u_{0j} = \text{const} > 0 \tag{1.6}$$

is imposed, then, in order that inequalities (1.6) not be violated during the stabilization, the initial perturbations $\mathbf{x}_0$ must be located in a sufficiently small neighborhood of the motion $\mathbf{x} = 0$. We specify this neighborhood by the inequality $\|\mathbf{x}_0\| < \rho$, where $\|\cdot\|$ is the Euclidean

*Prikl.Matem.Mekhan.,46,No.3,pp.366-373,1982 norm of the vector, and we estimate the quantity ρ.

If system (1.1) is autonomous, while expressions (1.4) have been defined in the region $|x_s| < l_s$ $(l_s > 0,\ s = 1, \ldots, n)$, then the hyper-surfaces $u_j^\circ(\mathbf{x}) = \pm u_{0j}$ pick out in the phase space a region G containing the point $\mathbf{x} = 0$ within it. By ∂G we denote the boundary of region G and we can write the required estimate of the quantity ρ as an inequality

$$\sup_{\|\mathbf{x}\| \leqslant \rho} V(\mathbf{x}) \leqslant \inf_{\mathbf{x} \in \partial G} V(\mathbf{x})$$

For a nonautonomous system (1.1) a similar estimate can be written as

$$\sup_{\|\mathbf{x}\| \leqslant \rho} V(t, \mathbf{x}) \leqslant \inf_{\mathbf{x} \in \partial G} V(t, \mathbf{x}), \quad t \geqslant t_0$$

(∂G is the boundary of the intersection of the domains $G(t)$ for $t \geqslant t_0$).

2. If the initial perturbations are arbitrary points of the domain $|x_s| < l_s$, then inequality (1.6) can be violated during the stabilization. The question arises on the preservation of the property of asymptotic stability of system (1.1) when constraints are imposed on the controls.

Let us first consider this problem for the autonomous system

$$\frac{dx_s}{dt} = X_s(\mathbf{x}) + \sum_{j=1}^{r} m_{sj}(\mathbf{x})\, u_j(\mathbf{x}) \quad (s = 1, 2, \ldots, n) \tag{2.1}$$

Let the constraints on the controls $u_j(\mathbf{x})$ be of the form $|u_j(\mathbf{x})| < u_{0j}(\mathbf{x})$, where $u_{0j}(\mathbf{x})$ are functions positive and continuous in domain (1.3). It is natural to define the controls $u_j(\mathbf{x})$ in such a way that they deviate as little as possible from the values $u_j^\circ(\mathbf{x})$, prescribed by relations (1.4). This requirement is satisfied by the definition

$$u_j = q_j u_j^\circ, \quad q_j = \begin{cases} 1, & |u_j^\circ| < u_{0j} \\ u_{0j}|u_j^\circ|^{-1}, & |u_j^\circ| \geqslant u_{0j} \end{cases} \tag{2.2}$$

For a control of this form the right-hand sides of Eqs.(2.1) are continuous in region (1.3). Therefore, the time derivative of the Liapunov function (Sect.1) relative to Eqs.(2.1) is defined and continuous in this region and has the form

$$\frac{dV}{dt} = W(\mathbf{x}) = \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} \sum_{j=1}^{r} m_{sj} u_j$$

Since

$$\sum_{s=1}^{n} m_{sj} \frac{\partial V}{\partial x_s} = -2 \sum_{i=1}^{r} \beta_{ij} u_i^\circ \quad (j = 1, \ldots, r)$$

with due regard to expression (2.2) we can write

$$\frac{dV}{dt} = W(\mathbf{x}) - 2\Sigma^\circ = W(\mathbf{x}) - 2\Sigma_q^\circ, \quad \Sigma^\circ = \sum_{i,j=1}^{r} \beta_{ij} u_i^\circ u_j^\circ, \quad \Sigma_q^\circ = \sum_{i,j=1}^{r} \beta_{ij} q_j u_i^\circ u_j^\circ \tag{2.3}$$

If $W(\mathbf{x})$ is a negative-definite function, then the function specified by relation (2.3) is also negative-definite. Therefore, the unperturbed motion of system (2.1), (2.2) is asymptotically stable. This conclusion can be extened to nonautonomous stabilizable systems for which the function $W(\mathbf{x})$ is negative-definite.

Let us now consider the case when the function $W(\mathbf{x})$ is identically zero. If the quadratic form Σ° is positive definite relative to u_j°, then the test for the asymptotic stability of the unperturbed motion of system (2.1) is the absence of whole trajectories in the manifold M^* defined by the equation $\Sigma_q^\circ = 0$. It is clear that if the quadratic form is positive-definite relative to u_j° $(j = 1, \ldots, r)$ for all $\mathbf{x}$ from the region $|x_s| < l_s$, then the manifold M^* coincides with the manifold M defined by Eq.(1.5) with $W(\mathbf{x}) \equiv 0$. A test for the positive definiteness of this form is the Sylvester criterion applied to the symmetric matrix with elements $\beta_{ij}(q_i + q_j)$ $(i, j = 1, \ldots, r)$. Assuming that the condition for the positive definiteness of the form Σ_q is fulfilled, we can conclude that manifold M^* lies inside region G. But in this region there are no constraints on system (2.1). Therefore, M^* cannot contain integral trajectories of system (2.1). Thus, if a dynamic system admits of an optimal stabilization in the sense of the method in /1,2/ in a region G_m of the phase space, then in the presence of modulus-constraints on the control it remains stabilizable by controls of form (2.2) in this region and optimally stabilizable in some region $G \subset G_m$. If a dynamic system has parameters which can be used as the stabilizing forces and if in this connection no energy consumption is required in the saturation mode, i.e., when $u_{0j} = \text{const}$, then controls of form (2.2) are optimal in the sense defined.

Let us consider an example of the optimal stabilization of the stationary motion of a satellite relative to the center of mass, located at a triangular libration point of a two-body system. The rotational motion of the satellite is characterized by the Hamiltonian /3/

$$
\begin{aligned}
H^\circ = & \frac{1}{2\sin^2\theta}[(p_\psi - p_\varphi\cos\theta)\sin\varphi + p_\theta\cos\varphi\sin\theta]^2 + \\
& \frac{1}{2B\sin^2\theta}[(p_\psi - p_\varphi\cos\theta)\cos\varphi - p_\theta\sin\varphi\sin\theta]^2 + \frac{A}{2C}p_\varphi^2 - p_\psi - \\
& \frac{3}{2}\frac{A-B}{A+C}\chi A_{12}^2 - \frac{3}{2}\frac{A-C}{A+C}\chi A_{13}^2, \quad \chi = (\psi - \Lambda)^2 + \nu(\psi + \Lambda)^2 \\
& A_{12} = -\cos\psi\sin\varphi - \sin\psi\cos\varphi\cos\theta, \quad A_{13} = \sin\psi\sin\theta, \quad \Lambda = 60^\circ, \\
& \nu = m_2/m_1
\end{aligned}
\tag{2.4}
$$

Here A, B and C are the principal central inertia moments of the satellite, m_1, m_2 are the masses of bodies M_1, M_2, $p_\theta, p_\psi, p_\varphi$ are the canonic momenta conjugate to the satellite's generalized coordinates θ, ψ, φ relative to a rotating coordinate system $Gxyz$ connected with bodies M_1 and M_2 /3/. As the independent variable we take the quantity $\tau = nt$ (where t is time, n is the mean orbital motion). The canonic equations of motion admit of a two-parameter family of solutions (ψ_0 is a constant precession angle)

$$
\begin{aligned}
& \theta = \pi/2, \ \varphi = 0, \ \psi = \psi_0, \ p_\theta = 0, \ p_\psi = B/A, \ p_\varphi = 0 \\
& \psi_0 = \psi_{00} + k\pi/2 \ (k = 0, 1, 2, 3), \ \cos 2\psi_{00} = -(1+\nu)/(2\sqrt{1-\nu+\nu^2})
\end{aligned}
\tag{2.5}
$$

(for the Earth—Moon system $\psi_{00} = 60°18'25''$).

For one of the solutions (2.5) let the stability conditions be fulfilled, according to which the angle ψ_0 lies within the limits $60°$ to $90°$ when $B > C > A$ (and within the limits $150°$ to $180°$ when $B > A > C$) /3/. We set the problem of stabilizing the stationary motion selected by changing the satellite's moments of inertia, for example, by means of displacing the massive rods along its principal axes of inertia. Let $\mathbf{v} = (v_1, v_2, v_3)$ be the variation of the positions of the centers of mass of the rod relative to the stationary value. Then the Hamiltonian can be written as

$$
\begin{aligned}
& H(\theta, \psi, \varphi, p_\theta, p_\psi, p_\varphi) = H^\circ + \frac{1}{2\sin^2\theta}[(p_\psi - p_\varphi\cos\theta)\sin\varphi - \\
& \quad p_\theta\cos\varphi\sin\theta]^2 f_1(\mathbf{v}) - \frac{3}{2}\chi A_{12}^2 f_2(\mathbf{v}) - \frac{3}{2}\chi A_{13}^2 f_3(\mathbf{v}) \\
& f_1(\mathbf{v}) = (B_0 + \lambda_1 v_1 + \lambda_3 v_3 + m_1 v_1^2 + m_3 v_3^2)^{-1} - B_0^{-1} \\
& f_2(\mathbf{v}) = (A_0 - B_0 + \lambda_2 v_2 - \lambda_3 v_3 + m_2 v_2^2 - m_3 v_3^2) f^{-1}(v) - \\
& \quad (A_0 - B_0)/(A_0 + C_0) \\
& f_3(\mathbf{v}) = (A_0 - C_0 - \lambda_1 v_1 + \lambda_3 v_3 - m_1 v_1^2 + m_3 v_3^2) f^{-1}(v) - (A_0 - C_0)/ \\
& \quad (A_0 + C_0) \\
& f(\mathbf{v}) = A_0 + C_0 + \lambda_1 v_1 + 2\lambda_2 v_2 + \lambda_3 v_3 + m_1 v_1^2 + 2m_2 v_2^2 + m_3 v_3^2, \\
& \lambda_i = 2b_i m_i.
\end{aligned}
\tag{2.6}
$$

Here m_1, m_2, m_3 are the masses of the rods displaceable along the axes $0\xi', 0\eta', 0\varphi'$ respectively, of the satellite's natural coordinate system (a pair of like rods is located on each axis symmetrically relative to the satellite's center of mass); b_1, b_2, b_3 are the distances of the satellite's center of mass to, respectively, the centers of masses m_1, m_2, m_3; $A_0 = A, B_0 = B, C_0 = C$ for $v_1 = v_2 = v_3 = 0$. Under the relations $2A_0^2 - 2A_0B_0 - A_0C_0 - C_0^2 \neq 0$, necessarily fulfilled under the problem's conditions, there holds the inequality $\det[\partial(f_1, f_2, f_3)/\partial(v_1, v_2, v_3)] \neq 0$, so that we can consider the problem of optimal stabilization the stationary motion, having taken $u_i = f_i(\mathbf{v})$ as new controls in a sufficiently small neighborhood of zero. Taking motion (2.5) as the unperturbed one, we introduce new values of the coordinates and the momenta by the relations $\theta = \pi/2 + \xi_1$, $\psi = \psi_0 + \xi_2$, $\varphi = \xi_3$, $p_\theta = \eta_1$, $p_\psi = B_0/A_0 + \eta_2$, $p_\varphi = \eta_3$. For the equations of motion of the perturbed system

$$
\begin{aligned}
& d(\xi_1, \xi_2, \xi_3)/d\tau = \partial H_1/\partial(\eta_1, \eta_2, \eta_3) \\
& d(\eta_1, \eta_2, \eta_3)/d\tau = -\partial H_1/\partial(\xi_1, \xi_2, \xi_3)
\end{aligned}
\tag{2.7}
$$

the Hamiltonian is

$$
\begin{aligned}
& H_1(\xi, \eta) = H_1^\circ(\xi, \eta) + \sum_{i=1}^{3}\psi_i(\xi, \eta)u_i \\
& H_1^\circ(\xi, \eta) = H^\circ(\pi/2 + \xi_1, \psi_0 + \xi_2, \xi_3, \eta_1, B/A + \eta_2, \eta_3) + \text{const} =
\end{aligned}
$$

$$= \frac{1}{2} a_{11}\xi_1^2 + \frac{1}{2} a_{22}\xi_2^2 + \frac{1}{2} a_{33}\xi_3^2 + \frac{1}{2}\eta_1^2 + \frac{A}{2B}\eta_2^2 + \frac{A}{2C}\eta_3^2 + a_{13}\xi_1\xi_3 + \xi_1\eta_3 + a_{34}\xi_3\eta_1 + \cdots$$

$$a_{11} = 3\frac{B-A}{A+C}\Lambda_0^2 \sin^2\psi_0, \quad a_{22} = 3\frac{C-A}{A+C}[\Lambda_0^2\cos^2\psi_0 + 2(-\Lambda_1 + \nu\Lambda_2) \times$$

$$(\sin 2\psi_0 + (1+\nu)\sin^2\psi_0)], \quad a_{33} = (B/A)^2 + (B-A)\Lambda_0^2\cos^2\psi_0/(A+C)$$

$$a_{13} = -3(B-A)\Lambda_0^2 \sin 2\psi_0/[2(A+C)], \quad a_{34} = B/A - 1$$

$$\psi_1(\xi, \eta) = \frac{1}{2}\cos^{-2}\xi_1[(B/A + \eta_2 + \eta_3\sin\xi_1)\sin\xi_3 - \eta_1\cos\xi_1\cos\xi_3]^2$$

$$\psi_2(\xi, \eta) = -\frac{3}{2}\Lambda_0^2 A_{120}^2, \quad \psi_3(\xi, \eta) = -\frac{3}{2}\Lambda_0^2 A_{130}^2$$

$$\Lambda_0^2 = \Lambda_1^2 + \nu\Lambda_2^2, \ \Lambda_1 = \Lambda - \psi_0, \ \Lambda_2 = \Lambda + \psi_0$$

$$A_{120} = -\cos(\psi_0 + \xi_2)\sin\xi_3 + \sin(\psi_0 + \xi_2)\cos\xi_3\sin\xi_1, \quad A_{130} = \sin(\psi_0 + \xi_2)\cos\xi_1$$

Here and henceforth we omit the index zero in the notation of the moments of inertia. The dots denote summands of higher than second order in the variables ξ_i, η_i, while the explicitly written out quadratic part is assumed positive-definite.

The control's performance index can be taken as

$$I = \int_{t_0}^{\infty}\left[F(\xi[t], \eta[t]) + \sum_{i=1}^{3}\beta_i u_i^2[t]\right]dt, \quad \beta_i > 0$$

Taking H_1° as the Liapunov function for system (2.7) when $u_i = 0$, we set up the expression

$$B[H_1^\circ, \xi, \eta, \mathbf{u}] = \sum_{i=1}^{3}(a_i u_i + \beta_i u_i^2) + F(\xi, \eta)$$

$$a_1 = \frac{\partial H_1^\circ}{\partial \xi_1}\frac{\partial \psi_1}{\partial \eta_1} + \frac{\partial H_1^\circ}{\partial \xi_2}\frac{\partial \psi_1}{\partial \eta_2} + \frac{\partial H_1^\circ}{\partial \xi_3}\frac{\partial \psi_1}{\partial \eta_3} - \frac{\partial H_1^\circ}{\partial \eta_1}\frac{\partial \psi_1}{\partial \xi_1} - \frac{\partial H_1^\circ}{\partial \eta_3}\frac{\partial \psi_1}{\partial \xi_3} \tag{2.8}$$

$$a_k = -\frac{\partial H_1^\circ}{\partial \eta_1}\frac{\partial \psi_k}{\partial \xi_1} - \frac{\partial H_1^\circ}{\partial \eta_2}\frac{\partial \psi_k}{\partial \xi_2} - \frac{\partial H_1^\circ}{\partial \eta_3}\frac{\partial \psi_k}{\partial \xi_3}, \quad k = 2,3$$

From the expressions $\partial B/\partial u_i = 0, B[H_1^\circ, \xi, \eta, \mathbf{u}^\circ] = 0$ we determine the required controls and the function $F(\xi, \eta)$

$$u_i^\circ = -\frac{1}{2\beta_i}a_i \ (i = 1, 2, 3), \quad F(\xi, \eta) = \frac{1}{4}\sum_{i=1}^{3}\frac{a_i^2}{\beta_i} \tag{2.9}$$

The function $F(\xi, \eta)$ is positive-definite with respect to a_i, while, in general, it is sign positive with respect to the variables ξ_i, η_i.

In order to be convinced that the control (2.9) found indeed does stabilize the motion of the perturbed system, we verify the absence of integral trajectories of system (2.7) in the manifold defined by the equations

$$a_1 = A_{120}^{-1}a_2 = a_3 = 0 \tag{2.10}$$

(the factor $A_{120} \neq 0$ occurs in the expression for a_2). Relations (2.10) can be treated as a system of linear equations relative to $\partial H_1^\circ/\partial \eta_i \ (i = 1, 2, 3)$. In a neighborhood of the solution $\xi_i = \eta_i = 0$

$$\det\frac{\partial(\psi_1, \psi_2, \psi_3)}{\partial(\xi_1, \xi_2, \xi_3)}A_{120}^{-1} = 9\Lambda_0^2\sin^2\psi_0[(-\Lambda_1 + \nu\Lambda_2)\sin\psi_1 + \Lambda_0^2\cos\psi_0] + o(1) \neq 0$$

Therefore, the system has a unique solution. In particular,

$$\partial H_1^\circ/\partial \eta_1 = (A/B)\operatorname{ctg}\psi_0(a_{11}\xi_1 + a_{13}\xi_3 + \eta_3) + \cdots,$$

which does not correspond to the value of $\partial H_1^\circ/\partial \eta_1$ for the given Hamiltonian H_1^c. Consequently, manifold M does not contain integral trajectories (except the trajectories $\xi_i = \eta_i = 0$), while the control (2.9) is optimal in the sense of the method in /1,2/. Controls (2.9) are constrained because of the restricted telescoping of the rods. Since here $\beta_{ij} = 0$ for $i \neq j$, the manifolds M^* and M coincide and the controls of form (2.2) stabilize the stationary motion (2.5) for any initial conditions from the region of possible librational motion of the satellite.

When the dynamic system's stabilization must be effected by the application of external forces economical from the point of view of energy consumption the control can be: $u_j = u_j^\circ$ if $|u_1^\circ| < u_{01}, \ldots, |u_r^\circ| < u_{0r}$; $u_j = 0$ if $|u_i^\circ| \geqslant u_{0i}$ for at least one number $i = 1, 2, \ldots, r$. Such a control will be a stabilizing control for initial perturbations $\mathbf{x}_0$ which lie on the trajectories of the dynamic system in the absence of controls intersecting regions G, and will be optimal in the sense that it minimizes each summand in the sum

$$\sum_{i=1}^{\infty} \int_{t_{2i-1}}^{t_{2i}} \left[F(\mathbf{x}[t]) + \sum_{i,j=1}^{r} \beta_{ij} u_i[t] u_j[t] \right] dt$$

for the time intervals (t_{2i-1}, t_{2i}) on which $u_j = u_j^c$, while energy consumption is not required in the other time intervals.

3. In some cases, using Liapunov's second method, we can make an estimate, important in practice, of the time in which the system to be stabilized (or the asymptotically stable system) falls, from an arbitrary point $\mathbf{x}_0$ in the domain of attraction of the unperturbed motion, into a specified region G_0 containing this unperturbed motion. As a matter of fact, let $V(t, \mathbf{x})$ be a positive-definite Liapunov function admitting of an infinitesimal upper bound, whose time derivative relative to the equations of perturbed motion is a negative-definite function $W(t, \mathbf{x})$. Then for the prescribed region G_0 we can choose a positive number V_1 such that the region G bounded by the surface $V(t, \mathbf{x}) = V_1$ will lie in region G_0 when $t \geqslant t_0$.

We define a region G_m demarked by the surface $V(t, \mathbf{x}) = \mathbf{x}_0$ as time t varies on the interval $[t_0, \infty)$, or some majorant of such a region. In the region $G_m \setminus G_0$, $t \geqslant t_0$, we determine the minimum value h of the function $|W(t, \mathbf{x})|$. Let T be the time in which the function $V(t, \mathbf{x})$ decreases from the value V_0 to the value V_1; then there holds the relation

$$V_0 - V_1 = -\int_{t_0}^{t_0+T} W(t, \mathbf{x})\, dt \geqslant hT$$

from which we obtain the required estimate $T \leqslant (V_0 - V_1)/h$. In the case when the asymptotically stable system admits, in the domain of attraction, of a manifold M defined by the equation $dV/dt = 0$, the problem of estimating T becomes complicated. Let us solve this problem for the asymptotically stable motion of the second-order system

$$dx_1/dt = X_1(x_1, x_2), \quad dx_2/dt = X_2(x_1, x_2) \tag{3.1}$$

In order to make certain preliminary estimates we introduce the polar coordinates r and θ by means of the substitutions $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and we write Eqs. (3.2) in the new coordinates after expanding the right-hand sides in powers of r

$$dr/dt = R_0(\theta) + rR_1(\theta) + r^2R_2(\theta) + \ldots \tag{3.2}$$

$$d\theta/dt = \vartheta_0(\theta) + r\vartheta_1(\theta) + r^2\vartheta_2(\theta) + \ldots \tag{3.3}$$

We assume the power series to be convergent uniformly relative to $\theta \in [0, 2\pi]$ in the circle $r < r_m$ containing the region G_m. If in these series $\cos\theta$ (or $\sin\theta$) occurs to an even power, then after the change of variable

$$\sin\theta = z \ (\text{or} \ \cos\theta = z) \tag{3.4}$$

the coefficients $R_i(z), \vartheta_i(z)$ will be certain polynomials in z. Further, we shall reckon that z is a complex quantity.

By the problem's conditions integers k_1, k_2 exist for which the inequalities

$$|P_{k_1}(z)| = |R_0(z) + rR_1(z) + \ldots + r^{k_1} R_{k_1}(z)| > |r^{k_1+1} R_{k_1+1}(z) + \ldots| \tag{3.5}$$

$$|P_{k_2}(z)| = |\vartheta_0(z) + r\vartheta_1(z) + \ldots + r^{k_2} R_{k_2}(z)| > |r^{k_2+1} R_{k_2+1}(z) + \ldots|$$

are fulfilled for $|z| = 1$ for any $r < r_m$. According to the Rouche theorem we can conclude that the numbers of zeros of the functions $P_1(z) = R_0(z) + rR_1(z) + \ldots$, $P_2(z) = \vartheta_0(z) + r\vartheta_1(z) + \ldots$ in the region $|z| < 1$ equal, respectively, the numbers ν_1, ν_2 of zeros of the polynomials P_{k_1}, P_{k_2}. Hence, on the segment $|\operatorname{Re} z| \leqslant 1$ the numbers of zeros of the functions $P_1(z)$, $P_2(z)$ are estimated by the inequalities

$$n_1 \leqslant 2\nu_1 + 2, \quad n_2 = 2\nu_2 + 2 \tag{3.6}$$

If $\sin\theta$ and $\cos\theta$ occur in the right-hand sides of Eqs. (3.2), (3.3) both to even as well as to odd powers, then the change of variable (3.4) leads to the appearance in coefficients $R_i(z)$, $\vartheta_i(z)$ of the radical $\sqrt{1 - z^2}$. This complicates the choice of the numbers k_1, k_2 for expressions of form (3.5) in view of the slow convergence as $|\operatorname{Re} z| \to 1$ of the power series representing this radical. Therefore, we divide the domain $[0, 2\pi]$ of the variable θ into the intervals

$$(-\pi/4, \pi/4), \ (3\pi/4, 5\pi/4), \ (\pi/4, 3\pi/4), \ (5\pi/4, 7\pi/4)$$

On the first two intervals we define a change of variable $z = \sin\theta$, while on the second two intervals, $z = \cos\theta$. In both cases $|\operatorname{Re} z| < 1/\sqrt{2}$; therefore, the expansion of the radical

$\sqrt{1-z}$ converges rapidly and we can estimate the summands in form (3.5). In the latter case we obtain the estimates

$$n_1 \leqslant 2 \ (n_{11}+n_{12})+4, \ n_2 \leqslant 2\,(n_{21}+n_{22})+4 \tag{3.7}$$

for the numbers n_1, n_2 of zeros. Here n_{11}, n_{21} are the numbers of zeros of the polynomials $P_{k_1}(z)$, $P_{k_2}(z)$ under the change of variable $z=\sin\theta$ and n_{12}, n_{22} are the numbers of zeros of the polynomials P_{k_1}, $P_{k_2}(z)$ under the change $z=\cos\theta$ in the region $|z|<1$. We note that the derivative $d\theta/dt$ does not vanish in the region G_m if it does not vanish for $\theta=0, \pm\pi/2, \pi$, $n_{21}=n_{22}=0$, $r<r_m$.

Let the phase trajectory γ pass at $t=0$ through the point $\mathbf{x}=\mathbf{x}_0$ of the boundary ∂G_m of region G_m. If in region G_m the derivative $d\theta/dt$ does not vanish, then, obviously, for a single circuit around the origin the estimate

$$J_2=J\,(\partial G_m)+n_1\,(r_m-\rho\,(0,\ \ \partial G_0))\geqslant J\,(\gamma_1)\geqslant J_1=\max\,\{\rho\,(\mathbf{x}_0,\ \partial G_0),\ J\,(\partial G_0{}^*)\}$$

holds for the length $J(\gamma_1)$ of a piece of the trajectory γ_1. Here $\rho(A_1, A_2)$ is the distance between sets A_1 and A_2; $J(\partial G_m)$, $J(\partial G_0{}^*)$ are the lengths of the boundary of region G_m and of the boundary of the convex hull $G_0{}^*$ of region G_0. Then the time t^* for one circuit of the representative point around the origin satisfies the inequality

$$t_1=J_1/v_1\leqslant t^*\leqslant J_2/v_2=t_2$$

$$v_1=\max_{\mathbf{x}\in G_m\setminus G_0}\sqrt{X_1{}^2(\mathbf{x})+X_2{}^2(\mathbf{x})}$$

$$v_2=\min_{\mathbf{x}\in G_m\setminus G_0}\sqrt{X_1{}^2(\mathbf{x})+X_2{}^2(\mathbf{x})}$$

A piece of the trajectory γ_1 can intersect the curves M defined by the equation $dV/dt=0$. For a sufficiently small value of the number $\varepsilon>0$ these curves will be contained in a region G_ε bounded by the curves M_1 and M_2 defined by the equation $dV/dt=-\varepsilon$. The set $G_\varepsilon\cap G_m\setminus G_0$ is the union of several connected regions. For the values $\theta=\theta_1,\theta_2,\ldots,\theta_m$ let the normal to curve M pass through the origin. Then these normals separate the set $G_\varepsilon\cap G_m\setminus G_0$ into m simply-connected domains $D_1, D_2,\ldots, D_m$. We remark that the curves M may not pass through all the domains D_i. For each of the domains D_i we can determine the minimum value of the phase velocity $v_{2i}>v_2>0$ and give an estimate of the largest length h_i of the phase trajectory, at least for a sufficiently small ε. The total time of motion of the representative point in the set $G_\varepsilon\cap G_m\setminus G_0$ for a single circuit around the origin does not exceed the quantity

$$\tau=\sum_{i=1}^{m}h_i/v_{2i}$$

while the time of motion on the remaining parts of curve γ_1 is not less than $\tau^*=t_1-\tau$.

Let us further assume that the phase trajectories in the region $G_m\setminus G_0$ are not tangent to curves M. Then $\tau\to 0$ as $\varepsilon\to 0$, and for a sufficiently small ε we have $\tau^*>0$. In time t^* the value of the Liapunov function decreases by the amount $\Delta V=V(\mathbf{x}_0)-V(\mathbf{x}(t^*))\geqslant\varepsilon\tau^*$. Therefore, in the whole time T of motion up to falling into region G_0 the representative point accomplishes no more than $n=[(V_0-V_1)/(\varepsilon\tau^*)]+1$ circuits around the origin. Consequently, $T\leqslant nt_2$. In case the derivative $d\theta/dt$ has n_2 zeros in the region $G_m\setminus G_0$, the estimate of time T obviously has the form $T\leqslant n\,(n_2+1)\,t_2$.

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